

Existence of global weak solutions for the Navier-Stokes-Vlasov-Boltzmann equations

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Abstract: The moderately thick spray can be described by a coupled system of equations consisting of the incompressible Navier-Stokes equations and the Vlasov-Boltzmann equation. We investigate this kind of mathematical model in this paper. In particular, we study the initial value problem of the Navier-Stokes-Vlasov-Boltzmann equations. The existence of global weak solutions is established by the weak convergence method. The interesting point of our main result is to handle the model with some breakup effects while the velocity of particles is in the whole space.

Key Words: The moderately thick spray, Breakup kernel, Navier-Stokes-Vlasov-Boltzmann equations, Existence, Weak solution.

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1 Introduction

A spray is a process in which drops dispersed in a gas. The sprays can help people to distribute material over a cross-section and to generate liquid surface area. There are various applications in different fields, the examples include but not limit fuel injectors for gasoline and Diesel engines, atomizers for jet engines, atomizers for injecting heavy fuel oil into combustion air in steam boiler injectors, and rocket engine injectors. It also has a big impact on crop yield, plant health, efficiency of pest control and of course, profitability. Here we will focus on the model of the moderately thick spray. In general, we refer the reader to [4] for physical backgrounds.

For the moderately thick spray, we can assume that the volume fraction occupied by the droplets is small enough to be neglected. Thus, we are able to apply the Vlasov-Boltzmann equation to model the liquid phase, which can be performed by the use of a particle density function. In particular, a function $f(t, x, \xi)$ denotes a number density of droplets of which

at time t and physical position x with velocity ξ , which is a solution of the following Vlasov-Boltzmann equation

$$f_t + \xi \cdot \nabla_x f + \nabla(f\mathfrak{F}) = Q(f, f), \quad (1.1)$$

where \mathfrak{F} is an acceleration resulting from the drag force exerted by the gas, and $Q(f, f)$ is an operator taking into account the complex phenomena happening at the level of the droplets (collisions, coalescences, breakup).

The incompressible Navier-Stokes equations could be used to describe the motion of gas when instigating the transport of sprays in the upper airways of the human lungs:

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla P - \mu \Delta u &= F_e, \\ \operatorname{div} u &= 0, \end{aligned} \quad (1.2)$$

where u is velocity of gas, and F_e denotes the external force with

$$F_e = -c \int_{\mathbb{R}^3} f \mathfrak{F} d\xi, \quad (1.3)$$

\mathfrak{F} denotes the acceleration, it is given by

$$\mathfrak{F} = -\frac{9\mu}{2\rho} \frac{\xi - u}{r^2},$$

μ is the viscosity constant of the incompressible Navier-Stokes equations, ρ is the density of gas, r is the radius of the droplet. We can assume that $\frac{9\mu}{2\rho r^2} = 1$ in the whole paper, thus

$$\mathfrak{F} = -(\xi - u). \quad (1.4)$$

One of the typical forms of the collision kernel is given by

$$Q(f, f) = -\lambda f(t, x, \xi) + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') d\xi'. \quad (1.5)$$

The constant $\lambda > 0$ is the breakup frequency. The kernel $T(\xi, \xi')$ is the probability of a change with respect to velocity from ξ' to ξ , and ξ is the velocity of individuals before the collision while ξ' is the velocity immediately after the collision. Given that a reorientation occurs, the probability function $T(\xi, \xi')$ is a non-negative function and after normalization we have

$$\int_{\mathbb{R}^3} T(\xi, \xi') d\xi = 1. \quad (1.6)$$

By the acceleration of the Vlasov-Boltzmann equations and the external force of the incompressible Navier-Stokes equations, the above equations (1.1)-(1.5) can be coupled with each other. As a result, it is a model for the moderately thick spray, namely the Navier-Stokes-Vlasov-Boltzmann equations:

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + \operatorname{div}_\xi((u - \xi)f) = -\lambda f + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') d\xi', \\ \partial_t u + u \cdot \nabla_x u + \nabla_x P - \mu \Delta_x u = -\int_{\mathbb{R}^3} (u - \xi) f d\xi, \\ \operatorname{div} u = 0, \end{cases} \quad (1.7)$$

with

$$\int_{\mathbb{R}^3} T(\xi, \xi') d\xi = 1.$$

The main goal of this paper is to investigate the existence of weak solutions globally in time t for equations (1.7) with the following initial data

$$(f, u)|_{t=0} = (f_0(x, \xi), u_0(x)), \quad x \in \mathbb{T}^3, \quad \xi \in \mathbb{R}^3, \quad (1.8)$$

and the initial data $f_0(x, \xi)$ is suitable decay condition as $|\xi| \rightarrow \infty$.

The mathematical analysis for spray models is very challengingly because the coupled term for unknowns that does not depend on the same set of variables. Concerning to the existence theory of global weak solution for such models, we should date back to late of 1990's. The first work in this field was done in [8] where the author proved the global weak existence and its large-time behavior for the Vlasov-Stokes equations. The existence theorem of weak solutions has been extended in [1, 3, 10, 12, 14], where the authors did not neglect the convection term and considered the Navier-Stokes equations, including incompressible and compressible ones; Among which, Anoschenko, Khruslov and Stephan obtained the global existence of weak solutions to the Navier-Stokes-Vlasov-Poisson system. In [5], the existence and uniqueness of global smooth solutions near equilibrium are proved under smallness conditions for the Navier-Stokes system coupled with the Vlasov-Fokker-Planck equation in 3D. In the mean time, there are a lot of works in hydrodynamic limits, we refer the reader to [6, 7, 11]. In these works, the authors rely on convergence methods, like compactness and relative entropy method, to investigate hydrodynamic limits. It is natural to treat the mathematical analysis for the models with the effects of collisions or the effects of breakup, but more challenging. In [9], Legar-Vasseur established the existence theory of the weak solutions for a system coupled by the Navier-Stokes equations and Vlasov-Blotzmann equation where they restricted the velocity ξ in a bounded domain. Benjelloun-Desvillettes-Moussa in [2] introduced the Navier-Stokes-Vlasov-Blotzman model for spray theory, and a typical operator is given when the droplets after breakup have the same velocities as before breakup. Finally, they established an existence result of weak solutions for a simple model which derived from the Navier-Stokes-Vlasov-Blotzman model. In particular, they assume that the aerosol is bidispersed, in other words, there are only two possible radius $r_1 > r_2$ exist for the droplets, and the result of the breakup of particles of radius r_1 are particles of radius r_2 . Under this assumption, the density function will split into as follows

$$f(t, x, \xi, r) = f_1(t, x, \xi) \delta_{r=r_1} + f_2(t, x, \xi) \delta_{r=r_2},$$

therefor, the Navier-Stokes-Vlasov-Blotzman model will reduced to the Navier-Stokes-Vlasov equations with damping terms. Motivated by the work of Yu [15], of particular interests in this paper is to establish the existence of weak solutions of the Navier-Stokes-Vlasov-Blotzman equations with a typical breakup operator given in [2], for the velocity $\xi \in \mathbb{R}^3$. However, the restriction of the same velocities is not necessary in our paper.

It is not necessary to assume that when the droplets after breakup have the same velocities as before breakup in this paper. In fact, we deduce the following ones from the conservation of kinetic energy,

$$|\xi|^2 = |\xi'|^2,$$

which implies

$$|\xi| = |\xi'|. \quad (1.9)$$

We will assume (1.9) in the whole paper. In other word, different from the work of [2], we only need to assume that the droplets after breakup have the same speeds as before breakup, not the same velocities.

As the same in the work of [9], we assume $T(\xi, \xi')$ satisfies a self-similarity property, namely,

$$T(\xi, \xi') = H(|\xi'|)T\left(\frac{\xi}{|\xi'|}, \frac{\xi'}{|\xi'|}\right), \text{ for some function } H(\cdot). \quad (1.10)$$

Note that, the smooth solution of the problem (1.7)-(1.8) satisfies the energy equality. In particular, we have energy inequality

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla u|^2 dx dt + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f|u - \xi|^2 d\xi dx dt \\ &= \frac{1}{2} \int_{\mathbb{T}^3} |u_0|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(1 + \frac{1}{2}|\xi|^2) d\xi dx, \end{aligned} \quad (1.11)$$

for any $T > 0$. Thus, it is natural to ask the initial data satisfies

$$\int_{\mathbb{T}^3} |u_0|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(1 + \frac{1}{2}|\xi|^2) d\xi dx < \infty. \quad (1.12)$$

Based on the energy equality (1.11), we give the concept of weak solution for the problem (1.7)-(1.8) in the following sense:

Definition 1.1. A pair (u, f) is called a global weak solution to the problem (1.7)-(1.8) if, for any $T > 0$, the following properties hold:

- $u \in L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3));$
- $f(t, x, \xi) \geq 0$, for any $(t, x, v) \in (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3;$
- $f \in L^\infty(0, T; L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3);$
- $|\xi|^3 f \in L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3));$
- for any test function $\varphi \in C^\infty([0, T] \times \mathbb{T}^3)$ with $\operatorname{div} \varphi = 0$, we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^3} (-u \cdot \varphi_t + (u \cdot \nabla) u \cdot \varphi + \nabla u : \nabla \varphi) dx dt \\ &= - \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(u - \xi) \cdot \varphi d\xi dx dt + \int_{\mathbb{T}^3} u_0 \cdot \varphi(0, x) dx; \end{aligned} \quad (1.13)$$

- for any test function $\phi \in C^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$, we have

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(\phi_t + \xi \cdot \nabla_x \phi + (u - \xi) \cdot \nabla_\xi \phi) d\xi dx dt = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0 \phi(0, x, \xi) d\xi dx \\
& + \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f \phi d\xi dx dt - \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \phi d\xi' d\xi dx dt;
\end{aligned} \tag{1.14}$$

- the energy inequality

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{T}^3} |u|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla_x u|^2 dx dt \\
& + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f |u - \xi|^2 d\xi dx dt \\
& \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(1 + \frac{1}{2} |\xi|^2) d\xi dx,
\end{aligned}$$

for any $t \in [0, T]$.

Our main results in this paper is as follows:

Theorem 1.1. *If the initial data satisfies $\operatorname{div} u_0 = 0$, (1.12) and $|\xi|^3 f_0 \in L^1(\mathbb{T}^3 \times \mathbb{R}^3)$, the probability function $T(\xi, \xi')$ satisfies (1.6) and (1.10), then there exists a global weak solution to the problem (1.7)-(1.8).*

Remark 1.1. *The same result in Theorem 1.1 holds for the initial boundary value problem (1.7)-(1.8), with the boundary condition $u = 0$ on $\partial\Omega$ and $f(t, x, \xi) = f(t, x, \xi^*)$ for $x \in \partial\Omega$, $\xi \cdot n(x) < 0$, where $\xi^* = \xi - 2(\xi \cdot n(x))n(x)$ is the specular velocity, $n(x)$ is the outward normal to Ω , and $\Omega \subset \mathbb{R}^3$ is a bounded domain. Meanwhile, the incompressible Navier-Stokes equations of system (1.7) can be replaced by the inhomogeneous ones, the extension of our result, in this context, is considered in the forthcoming paper [13].*

The interesting point of our main result is to handle the model with some breakup effects while the velocity of particles is in the whole space. As we mentioned before, we need to assume that the droplets after breakup have the same speeds as before breakup, not the same velocities. This is different from the work of [2]. A key observation of our proof is Lemma 2.2, which gives us some uniform control on density function $f(t, x, \xi)$. By the Fubini Theorem, we are able to show the following *a priori* estimate

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^p Q(f) d\xi dx = 0 \quad \text{for any } p \geq 1, \tag{1.15}$$

which allows us to obtain further bounds and compactness of smooth solutions. Thus, with a suitable approximation, a weak solution could be recovered. Our idea of the approximation is to construct an iteration for the kinetic part, and to adopt the Galerkin's method for the fluid part. However, the breakup operator in iteration is given by

$$- \lambda f_m^n + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_m^{n-1}(t, x, \xi') d\xi', \tag{1.16}$$

for any integers $n \geq 1$. Note that, $\{f_m^n\}$ is an increasing sequence with respect to n . This allows us to obtain the bounds on f_m^n and $\int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi$, which yields the weak stability. By the weak convergence method, the existence of weak solutions could be done.

Notations: In the following, C from line to line denote the generic positive constants depending on the initial data, T and the physical coefficients; $C(E, B)$ denotes the positive constants depending on the initial data, T , the physical coefficients and E, B .

We organize the rest of the paper as follows. In Section 2, we deduce a prior estimates, state some useful lemmas. In Section 3, we construct a smooth solution of an approximation scheme of (1.7)-(1.8). In Section 4, we recover the weak solutions from approximation by the weak convergence method.

2 *A priori* estimates and some useful lemmas

In this section, we derive a *a priori* estimates of the problem (1.7)-(1.8), which will help us to derive the weak stability of the solutions. Firstly, we derive the energy inequality for any smooth solutions of (1.7)-(1.8).

Lemma 2.1. *For any smooth solution (u, f) to the the problem (1.7)-(1.8), the following equality holds*

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla u|^2 dx dt + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f |u - \xi|^2 d\xi dx dt \\ &= \frac{1}{2} \int_{\mathbb{T}^3} |u_0|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(1 + \frac{1}{2} |\xi|^2) d\xi dx. \end{aligned} \quad (2.1)$$

Proof. Multiplying by u on both sides of (1.7)₂, and integrating over \mathbb{T}^3 , we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |u|^2 dx + \int_{\mathbb{T}^3} |\nabla u|^2 dx = - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(u - \xi) \cdot u d\xi dx. \quad (2.2)$$

Multiplying by $1 + \frac{1}{2} |\xi|^2$ on both sides of (1.7)₁, and integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f |u - \xi|^2 d\xi dx \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(u - \xi) \cdot u d\xi dx - \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(1 + \frac{1}{2} |\xi|^2) d\xi dx \\ & \quad + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') d\xi' (1 + \frac{1}{2} |\xi|^2) d\xi dx. \end{aligned} \quad (2.3)$$

By using the Fubini's theorem, (1.9) and (1.6), the last term on the right hand side of (2.3) can be estimated as follows

$$\begin{aligned} & \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') d\xi' (1 + \frac{1}{2} |\xi|^2) d\xi dx \\ &= \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') (1 + \frac{1}{2} |\xi'|^2) d\xi d\xi' dx \\ &= \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(t, x, \xi) (1 + \frac{1}{2} |\xi|^2) d\xi dx. \end{aligned} \quad (2.4)$$

Adding (2.2) to (2.3) and using (2.4) together, (2.1) follows. \square

To develop further estimates, we will rely on the following lemma.

Lemma 2.2 ([15]). *Assume $T(\xi, \xi')$ satisfies (1.6) and (1.10), then there exists a constant $K > 0$, such that*

$$\int_{\mathbb{R}^3} T(\xi, \xi') d\xi' \leq K < \infty. \quad (2.5)$$

In what follows, we denote

$$m_\alpha f(t, x) = \int_{\mathbb{R}^3} |\xi|^\alpha f d\xi, \quad \text{and} \quad M_\alpha f(t) = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^\alpha f dx d\xi,$$

here $\alpha \geq 0$ is a constant. Clearly,

$$M_\alpha f(t) = \int_{\mathbb{T}^3} m_\alpha f dx.$$

Lemma 2.3 ([8]). *Let $\beta > 0$ and g be a nonnegative function in $L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$, such that $m_\beta f(t, x) < +\infty$, for a.e. (t, x) . Then the following estimates holds for any $\alpha < \beta$:*

$$m_\alpha f(t, x) \leq C(\|f(t, x, \cdot)\|_{L^\infty(\mathbb{R}^3)} + 1)m_\beta f(t, x)^{\frac{\alpha+3}{\beta+3}}, \quad \text{a.e. } (t, x). \quad (2.6)$$

3 An approximation scheme

In this section, we construct smooth solutions of an approximation system. To that purpose, we define a finite dimensional space $X_m = \text{span}\{\phi_i\}_{i=1}^m$, where $\{\phi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{T}^3)$ is an orthonormal basis of $\{v \in L^2(\mathbb{T}^3) : \text{div} v = 0 \text{ in } \mathcal{D}'\}$. Define $Y_m = C([0, T]; X_m)$.

We propose the following approximation scheme associated with the Navier-Stokes-Boltzmann equations (1.7)

$$\begin{cases} \partial_t f_m + \xi \cdot \nabla_x f_m + \text{div}_\xi((\tilde{u} - \xi)f_m) = -\lambda f_m + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi', \\ \partial_t u_m + (\tilde{u} \cdot \nabla_x) u_m + \nabla_x P_m - \Delta_x u_m = - \int_{\mathbb{R}^3} (\tilde{u} - \xi) f_m d\xi, \\ \text{div} u_m = 0, \quad x \in \mathbb{T}^3, \xi \in \mathbb{R}^3, t \in (0, T), \end{cases} \quad (3.1)$$

with initial data

$$(f_m, u_m)|_{t=0} = (f_0^\epsilon(x, \xi), u_0^\epsilon(x)), \quad x \in \mathbb{T}^3, \xi \in \mathbb{R}^3, \quad (3.2)$$

where \tilde{u} is given in Y_m . The initial data f_0^ϵ and u_0^ϵ are C^∞ functions such that $f_0^\epsilon \rightarrow f_0$ strongly in $L^p(\mathbb{T}^3 \times \mathbb{R}^3)$, for all $p < \infty$, and weakly in $weak^*-L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$; f_0^ϵ has a compact support with respect to ξ in \mathbb{R}^3 , $M_3 f_0^\epsilon$ is uniformly bounded with respect to ϵ , and $u_0^\epsilon \rightarrow u_0$ strongly in $L^2(\mathbb{T}^3)$.

3.1 The weak solutions of kinetic part

The first step of solving (3.1)-(3.2) is to investigate the global existence of weak solutions of the following problem:

$$\begin{cases} \partial_t f_m + \xi \cdot \nabla_x f_m + \text{div}_\xi((\tilde{u} - \xi)f_m) = -\lambda f_m + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi', \\ f_m|_{t=0} = f_0^\epsilon. \end{cases} \quad (3.3)$$

To this end, we construct a sequence of solutions in n verifying

$$\begin{cases} \partial_t f_m^n + \xi \cdot \nabla_x f_m^n + \operatorname{div}_\xi((\tilde{u} - \xi) f_m^n) = -\lambda f_m^n + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_m^{n-1}(t, x, \xi') d\xi', \\ f_m^n|_{t=0} = f_0^\epsilon, \\ f_m^0 = 0. \end{cases} \quad (3.4)$$

The characteristic method gives us a solution of (3.4) for any given f_m^{n-1} . Indeed, when f_m^{n-1} is given, we are able to define the trajectories $x(\tau) = x(\tau, t, x, \xi)$ and $\xi(\tau) = \xi(\tau, t, x, \xi)$ with the following ODE systems

$$\begin{cases} \frac{dx}{d\tau}(\tau) = \xi(\tau), \\ \frac{d\xi}{d\tau}(\tau) = \tilde{u}(t, x(\tau)) - \xi(\tau), \\ x(t, t, x, \xi) = x, \\ \xi(t, t, x, \xi) = \xi. \end{cases}$$

Along the trajectories above, the solutions of (3.4) satisfies

$$\frac{d}{d\tau} f_m^n(\tau, x(\tau), \xi(\tau)) = (3 - \lambda) f_m^n(\tau, x(\tau), \xi(\tau)) + \lambda \int_{\mathbb{R}^3} T(\xi(\tau), \xi') f_m^{n-1}(\tau, x(\tau), \xi') d\xi'. \quad (3.5)$$

By the standard theory of ODE, there exists a smooth solution of (3.5) as follows

$$\begin{aligned} f_m^n(t, x, \xi) &= e^{(3-\lambda)t} f_0^\epsilon(x(0, t, x, \xi), \xi(0, t, x, \xi)) \\ &+ \lambda \int_0^t \int_{\mathbb{R}^3} e^{(3-\lambda)(t-\tau)} T(\xi(\tau, t, x, \xi), \xi') f_m^{n-1}(\tau, x(\tau, t, x, \xi), \xi') d\xi' d\tau. \end{aligned} \quad (3.6)$$

It is also a smooth solution of (3.4) for any given f_m^{n-1} . Thanks to (3.6), we find

$$f_m^n \geq 0, \text{ for all } n \geq 0,$$

and $\{f_m^n\}_{n=1}^\infty$ is an increasing sequence of measure functions with respect to n . For this proof, we refer the reader to [9].

Next we shall apply the compactness argument to recover the weak solutions of problem (3.3) by passing to the limits of f_m^n as n goes to infinity. To this end, we need to derive uniform estimates of f_m^n with respect to n . Note that, $\{f_m^n\}_{n=1}^\infty$ is an increasing nonnegative function. The Gronwall inequality yields the uniform estimates of f_m^n with respect to n .

The following Lemma 3.1 provides that f_m^n is bounded in $L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3))$, for any $p \geq 1$, with respect to $n > 0$.

Lemma 3.1. *For any $n \geq 0$ and fixed $m > 0$, $f_m^n(t, x, \xi)$ is bounded in*

$$L^\infty(0, T; L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3)),$$

and hence

$$f_m^n(t, x, \xi) \text{ is bounded in } L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3)), \text{ for any } p \geq 1. \quad (3.7)$$

Proof. Thanks to Lemma 2.2, we derive the following bound from (3.6)

$$\begin{aligned}\|f_m^n(t, x, \xi)\|_{L^\infty} &\leq e^{(3-\lambda)|T|} \|f_0^\epsilon\|_{L^\infty} + \lambda e^{(3-\lambda)|T|} K \int_0^t \|f_m^{n-1}(\tau, x, \xi)\|_{L^\infty} d\tau \\ &\leq e^{(3-\lambda)|T|} \|f_0\|_{L^\infty} + \lambda e^{(3-\lambda)|T|} K \int_0^t \|f_m^n(\tau, x, \xi)\|_{L^\infty} d\tau,\end{aligned}\tag{3.8}$$

where we used $\{f_m^n\}$ is an increasing sequence with respect to n .

Applying the Gronwall inequality to (3.8), one obtains

$$\|f_m^n\|_{L^\infty} \leq e^{(3-\lambda)|T|} \|f_0\|_{L^\infty} (1 + \lambda e^{(3-\lambda)|T|} K t e^{\lambda e^{(3-\lambda)|T|} K t}), \text{ for any } 0 \leq t \leq T.$$

Next, integrating (3.5) over $(0, t) \times \mathbb{T}^3 \times \mathbb{R}^3$, we find

$$\begin{aligned}\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(t, x, \xi) d\xi dx &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(x(0, t, x, \xi), \xi(0, t, x, \xi)) d\xi dx \\ &\quad + (3 - \lambda) \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx d\tau \\ &\quad + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} T(\xi(\tau), \xi') f_m^{n-1}(t, x(\tau), \xi') d\xi' d\xi dx d\tau,\end{aligned}$$

which implies

$$\begin{aligned}\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(t, x, \xi) d\xi dx &\leq \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(x(0, t, x, \xi), \xi(0, t, x, \xi)) d\xi dx \\ &\quad + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi(\tau), \xi') f_m^{n-1}(\tau, x(\tau), \xi') d\xi' d\xi dx d\tau \\ &\quad + 3 \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx d\tau \\ &\leq \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(x, \xi) d\xi dx + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi(\tau), \xi') f_m^{n-1}(\tau, x(\tau), \xi') d\xi' d\xi dx d\tau \\ &\quad + 3 \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx d\tau,\end{aligned}\tag{3.9}$$

where we used facts $T(\xi, \xi') \geq 0$, $f_m^n \geq 0$ for all $n \geq 0$, and $\lambda > 0$.

Thanks to (1.6), the Fubini's Theorem, we have

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(t, x, \xi) d\xi dx \leq \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0(x, \xi) d\xi dx + (\lambda + 3) \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(\tau, x, \xi) d\xi dx d\tau.$$

Applying the Gronwall inequality again, this yields

$$\|f_m^n\|_{L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3))} \leq \lambda e^{(3-\lambda)|T|} K (1 + (\lambda + 3) t e^{(\lambda+3)t}), \text{ for any } 0 \leq t \leq T.$$

□

Next we shall show that $\int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi'$ is bounded in $L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3))$, for any $p \geq 1$ in the following lemma.

Lemma 3.2. For any $n \geq 0$, $\int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi'$ is bounded in

$$L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3)), \text{ for any } p \geq 1.$$

Proof. We derive the following bound from (2.5) and (3.7):

$$\left\| \int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi' \right\|_{L^\infty} \leq \|f_m^n\|_{L^\infty} \int_{\mathbb{R}^3} T(\xi, \xi') d\xi' \leq C. \quad (3.10)$$

In the meanwhile, by (1.6), (3.7) and Fubini's theorem, we find

$$\begin{aligned} & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi' \right| d\xi dx \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi' d\xi dx \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n d\xi dx \leq C, \end{aligned} \quad (3.11)$$

here $C > 0$ depends only on the initial data, λ and T . This completes the proof of Lemma 3.2. \square

In order to pass to the limit as n goes to large, we need the bounds of $\int_{\mathbb{R}^3} f_m^n d\xi$, $\int_{\mathbb{R}^3} \xi f_m^n d\xi$ and $\int_{\mathbb{R}^3} |\xi|^2 f_m^n d\xi$ in the following lemma.

Lemma 3.3. For any $n \geq 0$, if

$$M_k f_0^\epsilon = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_0^\epsilon d\xi dx < +\infty,$$

for some $k \geq 1$, then the following estimates hold

$$\left\| \int_{\mathbb{R}^3} f_m^n d\xi \right\|_{L^\infty(0, T; L^{\frac{3+k}{3}}(\mathbb{T}^3))} \leq C(\lambda, T, M_k f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty}, \tilde{u}), \quad (3.12)$$

$$\left\| \int_{\mathbb{R}^3} \xi f_m^n d\xi \right\|_{L^\infty(0, T; L^{\frac{3+k}{4}}(\mathbb{T}^3))} \leq C(\lambda, T, M_k f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty}, \tilde{u}), \quad (3.13)$$

and

$$\left\| \int_{\mathbb{R}^3} |\xi|^2 f_m^n d\xi \right\|_{L^\infty(0, T; L^{\frac{3+k}{5}}(\mathbb{T}^3))} \leq C(\lambda, T, M_k f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty}, \tilde{u}). \quad (3.14)$$

Proof. Multiplying by $|\xi|^k$ on both sides of (3.5), then integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, for $k \geq 1$, we have

$$\begin{aligned} & \frac{d}{d\tau} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx \\ &= \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi(\tau), \xi') |\xi|^k f_m^{n-1}(\tau, x(\tau), \xi') d\xi' d\xi dx + 3 \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx \\ &\leq \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^{n-1}(\tau, x(\tau), \xi) d\xi dx + 3 \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(\tau, x(\tau), \xi(\tau)) d\xi dx. \end{aligned} \quad (3.15)$$

Taking integration with respect to time τ over $(0, t)$, then

$$\begin{aligned}
& \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx \\
& \leq \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi dx dt + 3 \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi dx dt \\
& + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_0^\epsilon d\xi dx.
\end{aligned} \tag{3.16}$$

Applying the Gronwall inequality to (3.16), there exists constant $K > 0$ such that,

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx \leq K e^{Kt},$$

for any $m, n > 0$, and $k \geq 1$. Here K depends on the initial data. This estimate, with Lemma 2.3 respectively, yields (3.12), (3.14) and (3.13). \square

Similar, we can show the following lemma:

Lemma 3.4. *Assume $f_0^\epsilon \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)$ and $|\xi|^k f_0^\epsilon \in L^1(\mathbb{T}^3 \times \mathbb{R}^3)$. If $f_m^n \in L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)$, then*

$$M_k f_m^n(t) \leq C_{N,T} \left((M_k f_0^\epsilon)^{\frac{1}{3+k}} + (\|f_m^n\|_{L^\infty} + 1) \|\tilde{u}\|_{L^p(0,T;L^{3+k}(\mathbb{T}^3))} \right)^{3+k},$$

for all $0 \leq t \leq T$.

Proof. Multiplying by $|\xi|^k$ on both sides of (3.4), one finds

$$\begin{aligned}
& \frac{d}{d\tau} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx - k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (\tilde{u} - \xi) f_m^n |\xi|^{k-1} d\xi dx \\
& = -\lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') |\xi|^k f_m^{n-1}(t, x, \xi') d\xi' d\xi dx,
\end{aligned}$$

which yields

$$\begin{aligned}
& \frac{d}{d\tau} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx + k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi dx + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx \\
& \leq k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\tilde{u}| |\xi|^{k-1} f_m^n d\xi dx + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') |\xi|^k f_m^{n-1}(t, x, \xi') d\xi' d\xi dx, \\
& \leq k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\tilde{u}| |\xi|^{k-1} f_m^n d\xi dx + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^{n-1}(t, x, \xi) d\xi dx \\
& \leq k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\tilde{u}| |\xi|^{k-1} f_m^n d\xi dx + \lambda \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx
\end{aligned}$$

where we have used f_m^n is an increasing sequence with respect to n . Thus, we have

$$\begin{aligned}
& \frac{d}{d\tau} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n(t, x, \xi) d\xi dx + k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi dx \\
& \leq k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\tilde{u}| |\xi|^{k-1} f_m^n d\xi dx.
\end{aligned} \tag{3.17}$$

Using Hölder's inequality, the right hand side of (3.17) can be estimated as follows:

$$k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^{k-1} f_m^n |\tilde{u}| d\xi dx \leq k \|\tilde{u}\|_{L^q(\mathbb{T}^3)} \|m_{k-1} f_m^n\|_{L^{q'}(\mathbb{T}^3)},$$

here $\frac{1}{q} + \frac{1}{q'} = 1$. Let $R > 0$ be fixed, then we have

$$m_{k-1} f_m^n(t, x) \leq C \|f_m^n(t)\|_{L^\infty} R^{k+2} + \frac{1}{R} \int_{|\xi| > R} |\xi|^k f_m^n d\xi,$$

taking $R = (\int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi)^{\frac{1}{k+3}}$ and $q = k + 3$, we get

$$k \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^{k-1} f_m^n |\tilde{u}| d\xi dx \leq Ck \|w(t)\|_{L^{k+3}(\mathbb{T}^3)} (\|f_m^n(t)\|_{L^\infty} + 1) \left(\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^k f_m^n d\xi dx \right)^{\frac{k+2}{k+3}},$$

substituting this into (3.17), we complete the proof of Lemma 3.4. \square

Thus, we are ready to show the following existence of weak solutions of approximation (3.3).

Lemma 3.5. *For any $T > 0$ and fixed $m > 0$, there exists a weak solution of approximation (3.3) in the following sense:*

$$\begin{aligned} & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(t) \phi(t, x, \xi) d\xi dx - \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(\phi_t + \xi \cdot \nabla_x \phi + (\tilde{u} - \xi) \cdot \nabla_\xi \phi) d\xi dx ds \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon \phi(0, x, \xi) d\xi dx + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m \phi d\xi dx ds \\ & \quad - \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') \phi d\xi' d\xi dx ds. \end{aligned} \quad (3.18)$$

In particular, the solution satisfies the following bounds:

$$\|f_m(t, x, \xi)\|_{L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C(\lambda, T, \|f_0^\epsilon\|_{L^\infty \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)}), \text{ for any } p \geq 1, \quad (3.19)$$

$$\left\| \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi' \right\|_{L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C(\lambda, T, \|f_0^\epsilon\|_{L^\infty \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)}), \text{ for any } p \geq 1, \quad (3.20)$$

$$\left\| \int_{\mathbb{R}^3} f_m d\xi \right\|_{L^\infty(0, T; L^2(\mathbb{T}^3))} \leq C(\lambda, T, M_3 f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty}, \tilde{u}), \quad (3.21)$$

$$\left\| \int_{\mathbb{R}^3} \xi f_m d\xi \right\|_{L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{T}^3))} \leq C(\lambda, T, M_3 f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty}, \tilde{u}), \quad (3.22)$$

and

$$\left\| \int_{\mathbb{R}^3} |\xi|^2 f_m d\xi \right\|_{L^\infty(0, T; L^{\frac{6}{5}}(\mathbb{T}^3))} \leq C(\lambda, T, M_3 f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty}, \tilde{u}). \quad (3.23)$$

Proof. In order to pass to the limits as $n \rightarrow \infty$, we investigate the convergence of operator $\int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi'$ in the suitable space. To this end, we need the following convergence

as n goes to infinity. Since f_m^n is an increasing sequence with respect to n , then by Lemma 3.1 and monotone convergence theorem, as $n \rightarrow \infty$, we have

$$\begin{aligned} f_m^n &\rightharpoonup f_m, \text{ weakly(*) in } L^\infty(0, T; L^q(\mathbb{T}^3 \times \mathbb{R}^3)), \text{ for any } q > 1; \\ f_m^n &\rightarrow f_m, \text{ strongly in } L^p(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3)), \text{ for any } 1 \leq p < \infty; \end{aligned} \quad (3.24)$$

Meanwhile, for any smooth test function $\psi(x)$, we find

$$\begin{aligned} &\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} T(\xi, \xi') f_m^{n-1}(t, x, \xi') d\xi' - \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi' \right) \psi(x) d\xi dx \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') (f_m^{n-1}(t, x, \xi') - f_m(t, x, \xi')) \psi(x) d\xi' d\xi dx \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (f_m^{n-1}(t, x, \xi) - f_m(t, x, \xi)) \psi(x) d\xi dx \rightarrow 0 \end{aligned} \quad (3.25)$$

as $n \rightarrow \infty$, thanks to (3.24). By Lemma 3.2, this yields

$$\int_{\mathbb{R}^3} T(\xi, \xi') f_m^n(t, x, \xi') d\xi' \rightharpoonup \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi' \text{ weakly(*) in } L^\infty(0, T; L^q(\mathbb{T}^3 \times \mathbb{R}^3)), \quad (3.26)$$

for any $q > 1$.

On the other hand, from (3.12), (3.13) and (3.14), we deduce the following convergence

$$\int_{\mathbb{R}^3} f_m^n d\xi \rightharpoonup \int_{\mathbb{R}^3} f_m d\xi \text{ weakly(*) in } L^\infty(0, T; L^2(\mathbb{T}^3)), \quad (3.27)$$

$$\int_{\mathbb{R}^3} \xi f_m^n d\xi \rightharpoonup \int_{\mathbb{R}^3} \xi f_m d\xi \text{ weakly(*) in } L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{T}^3)), \quad (3.28)$$

and

$$\int_{\mathbb{R}^3} |\xi|^2 f_m^n d\xi \rightharpoonup \int_{\mathbb{R}^3} |\xi|^2 f_m d\xi \text{ weakly(*) in } L^\infty(0, T; L^{\frac{6}{5}}(\mathbb{T}^3)), \quad (3.29)$$

as $n \rightarrow \infty$.

Since f_m^n given by (3.6) is a smooth solution of problem (3.4), it satisfies the following weak formulation

$$\begin{aligned} &\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(t, x, \xi) \phi(t, x, \xi) d\xi dx - \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n(\phi_t + \xi \cdot \nabla_x \phi + (\tilde{u} - \xi) \cdot \nabla_\xi \phi) d\xi dx ds \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon \phi(0, x, \xi) d\xi dx + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n \phi d\xi dx ds \\ &\quad - \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_m^{n-1}(t, x, \xi') \phi d\xi' d\xi dx ds, \end{aligned} \quad (3.30)$$

for any test function $\phi \in C^\infty([0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3)$.

Letting n tend to infinity in (3.30), and using (3.24), (3.26), (3.27) and (3.28), one obtains

$$\begin{aligned} &\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(t) \phi(t, x, \xi) d\xi dx - \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(\phi_t + \xi \cdot \nabla_x \phi + (\tilde{u} - \xi) \cdot \nabla_\xi \phi) d\xi dx ds \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon \phi(0, x, \xi) d\xi dx + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m \phi d\xi dx ds \\ &\quad - \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') \phi d\xi' d\xi dx ds. \end{aligned}$$

Here we should give a remark that the solution f_m satisfies the following bounds

$$\|f_m(t, x, \xi)\|_{L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \text{ for any } p \geq 1, \quad (3.31)$$

$$\left\| \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi' \right\|_{L^\infty(0, T; L^p(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \text{ for any } p \geq 1, \quad (3.32)$$

$$\left\| \int_{\mathbb{R}^3} f_m d\xi \right\|_{L^\infty(0, T; L^2(\mathbb{T}^3))} \leq C, \quad (3.33)$$

$$\left\| \int_{\mathbb{R}^3} \xi f_m d\xi \right\|_{L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{T}^3))} \leq C, \quad (3.34)$$

and

$$\left\| \int_{\mathbb{R}^3} |\xi|^2 f_m d\xi \right\|_{L^\infty(0, T; L^{\frac{6}{5}}(\mathbb{T}^3))} \leq C, \quad (3.35)$$

where C are positive constants depend only on $\lambda, T, M_3 f_0^\epsilon$ and $\|f_0^\epsilon\|_{L^\infty \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)}$. \square

3.2 The Navier-Stokes part

In this subsection, we shall study the solution of the Navier-Stokes part, and the energy inequality for the whole approximation (3.1). First, we are able to view the right hand side of (3.1)₂ as a external force of the Navier-Stokes equations. From (3.21) and (3.22), the right hand side of (3.1)₂ can be estimated as follows:

$$\begin{aligned} & \left\| - \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right\|_{L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{T}^3))} \\ &= \left\| -\tilde{u} \int_{\mathbb{R}^3} f_m d\xi + \int_{\mathbb{R}^3} \xi f_m d\xi \right\|_{L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{T}^3))} \\ &\leq C(m, \lambda, T, M_3 f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)}, \|\tilde{u}\|_{L^\infty(0, T; L^2(\mathbb{T}^3))}). \end{aligned} \quad (3.36)$$

Thus, we can apply the classical theory of Navier-Stokes equations to solve (3.1)₂ when f_m is a solution of (3.3).

Next we consider the following weak formulation of the Navier-Stokes equation (3.1)₂:

$$\begin{aligned} & \int_{\mathbb{T}^3} [\partial_t u_m \cdot \varphi + (\tilde{u} \cdot \nabla) u_m \cdot \varphi + \nabla u_m : \nabla \varphi] dx \\ &= - \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right) \cdot \varphi dx, \end{aligned} \quad (3.37)$$

where $\varphi \in X_m$. Since X_m is a finite dimensional space, we can write u_m as follows

$$u_m = \sum_{i=1}^m \alpha_{im}(t) \varphi_i.$$

By the standard Galerkin method, approximation (3.1)₂ yields the following ODE:

$$\begin{aligned} \frac{d}{dt} \alpha_{im}(t) &= - \int_{\mathbb{T}^3} (\tilde{u} \cdot \nabla \varphi_j) \cdot \varphi_i dx \alpha_{jm}(t) - \int_{\mathbb{T}^3} \nabla \varphi_i : \nabla \varphi_j dx \alpha_{jm}(t) \\ &\quad - \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right) \cdot \varphi_i dx. \end{aligned} \quad (3.38)$$

From (3.36), then by using the classical ODE theory, there exists a unique solution $\alpha_{im}(t)$ ($i = 1, \dots, m$) of (3.38) for any $0 \leq t \leq T_m$, where $0 < T_m \leq T$. This gives us a unique solution $u_m \in Y_m$ of weak formulation (3.37) for any $0 \leq t \leq T_m$.

Next, we can derive the energy inequality for u_m . Indeed, taking φ_i for (3.37) and multiplying the equation by α_{im} , then summing the resulting equality from $i = 1$ to m , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_{\mathbb{T}^3} |\nabla u_m|^2 dx \\ &= - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) \cdot u_m d\xi dx \\ &\leq \|u_m\|_{L^\infty(\mathbb{T}^3)} \cdot \left\| \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right\|_{L^1(\mathbb{T}^3)} \\ &\leq \frac{1}{2} \int_{\mathbb{T}^3} |\nabla u_m|^2 dx + C(m) \left\| \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right\|_{L^{\frac{3}{2}}(\mathbb{T}^3)}^2, \end{aligned}$$

where we used a fact that all norms in X_m are equivalent. This yields

$$\begin{aligned} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_0^t \int_{\mathbb{T}^3} |\nabla u_m|^2 dx ds &\leq \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx \\ &+ C(m) \int_0^t \left\| \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right\|_{L^{\frac{3}{2}}(\mathbb{T}^3)}^2 ds. \end{aligned} \quad (3.39)$$

With the help of (3.36), this gives us the following estimates

$$\begin{aligned} \|u_m\|_{L^2(\mathbb{T}^3)} &\leq M < \infty \quad \text{for any } t \in [0, T_m], \\ \|\nabla u_m\|_{L^2(0, T; L^2(\mathbb{T}^3))} &\leq M. \end{aligned} \quad (3.40)$$

Now we define a convex set

$$\mathbf{A} := \{\tilde{u} \in C([0, T_m]; X_m) : \sup_{0 \leq t \leq T_m} \|\tilde{u}\|_{L^2(\mathbb{T}^3)} \leq M, \operatorname{div} \tilde{u} = 0\},$$

and a map $S : \mathbf{A} \rightarrow \mathbf{A}$ such that $u_m := S(\tilde{u})$. We shall apply the Schauder's fixed point theorem to show that the operator S has a fixed point in the following lemma.

Lemma 3.6. *The operator S has a fixed point in \mathbf{A} , that is, there is a point $u_m \in \mathbf{A}$ such that $Su_m = u_m = \tilde{u}$.*

Proof. Thanks to (3.40), u_m is bounded in the set \mathbf{A} . Meanwhile, taking φ in (3.37) to be φ_i , multiplying the equation by $\alpha'_{im}(t)$, then summing the resulting equality from $i = 1$ to m , we have

$$\begin{aligned} \int_{\mathbb{T}^3} |\partial_t u_m|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\nabla u_m|^2 dx &= - \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right) \cdot \partial_t u_m dx \\ &\quad - \int_{\mathbb{T}^3} ((\tilde{u} \cdot \nabla) u_m) \cdot \partial_t u_m dx \\ &\leq \|\partial_t u_m\|_{L^\infty(\mathbb{T}^3)} \cdot \|\tilde{u}\|_{L^2(\mathbb{T}^3)} \cdot \|\nabla u_m\|_{L^2(\mathbb{T}^3)} + \|\partial_t u_m\|_{L^\infty(\mathbb{T}^3)} \cdot \left\| \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right\|_{L^1(\mathbb{T}^3)} \\ &\leq \frac{1}{2} \int_{\mathbb{T}^3} |\partial_t u_m|^2 dx + C(m) \left\| \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) d\xi \right\|_{L^1(\mathbb{T}^3)}^2 + C(m) \|\tilde{u}\|_{L^2(\mathbb{T}^3)}^2 \|\nabla u_m\|_{L^2(\mathbb{T}^3)}^2. \end{aligned}$$

Note that all norms in X_m are equivalent to each other, (3.36) and (3.40), one obtains that

$$\int_0^t \int_{\mathbb{T}^3} |\partial_t u_m|^2 dx ds \leq C(m, \lambda, T, M_3 f_0^\epsilon, \|f_0^\epsilon\|_{L^\infty \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)}, \|\tilde{u}\|_{L^\infty(0, T; L^2(\mathbb{T}^3))}), \quad (3.41)$$

for any $t \in [0, T_m]$. Thanks to (3.40)₂ and (3.41), the Aubin-Lions Lemma, $u_m = S(\tilde{u})$ is compact in \mathbf{A} . On the other hand, it is easy to verify that S is sequentially continuous, see [9, 12] for the details. Thus the Schauder's fixed point theorem gives that S has a fixed point u_m in \mathbf{A} . \square

We have shown that there exists (f_m, u_m) on a small time interval $[0, T_m]$. In an effort to extend T_m to T , we rely on the following uniform bounds on u_m .

Lemma 3.7. *For any $t \in [0, T_m]$, we have*

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m (1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} |\nabla_x u_m|^2 dx ds \\ & + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds \\ & \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx. \end{aligned} \quad (3.42)$$

Proof. Taking φ in (3.37) to be φ_i and multiplying the equation by α_{im} , the summing from $i = 1$ to m , we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_0^t \int_{\mathbb{T}^3} |\nabla u_m|^2 dx ds \\ & = \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx - \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m (\tilde{u} - \xi) \cdot u_m d\xi dx ds. \end{aligned} \quad (3.43)$$

Note that $\tilde{u} = u_m$ from Lemma 3.6. We are able to use u_m to replace \tilde{u} in energy equality of Navier-Stokes part (3.43) on a small time $[0, T_m]$, thus

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_0^t \int_{\mathbb{T}^3} |\nabla u_m|^2 dx ds \\ & = \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx - \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m (u_m - \xi) \cdot u_m d\xi dx ds. \end{aligned} \quad (3.44)$$

Note that, f_m^n given by (3.6) is a smooth solution to the problem (3.4). We can multiply $1 + \frac{1}{2} |\xi|^2$ on both sides of (3.4) and integrate them to have the following equality

$$\begin{aligned} & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n (1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n |\tilde{u} - \xi|^2 d\xi dx ds \\ & = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n (\tilde{u} - \xi) \cdot \tilde{u} d\xi dx ds - \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n (1 + \frac{1}{2} |\xi|^2) d\xi dx ds \\ & + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_m^{n-1}(t, x, \xi') d\xi' (1 + \frac{1}{2} |\xi|^2) d\xi dx ds \\ & = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n (\tilde{u} - \xi) \cdot \tilde{u} d\xi dx ds - \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^n (1 + \frac{1}{2} |\xi|^2) d\xi dx ds \\ & + \lambda \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m^{n-1}(t, x, \xi) (1 + \frac{1}{2} |\xi|^2) d\xi dx ds, \end{aligned} \quad (3.45)$$

where we used the Fubini's theorem, (1.9) and (1.6) in the last equality of (3.45). Next, by the convergence (3.24)-(3.29), we are able to pass to the limits in (3.45) as $n \rightarrow \infty$. In fact, we can recover the following inequality

$$\begin{aligned} & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |\tilde{u} - \xi|^2 d\xi dx ds \\ & \leq \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(\tilde{u} - \xi) \cdot \tilde{u} d\xi dx ds. \end{aligned} \quad (3.46)$$

Applying Lemma 3.6 to (3.46), one obtains

$$\begin{aligned} & \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds \\ & \leq \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^t \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(u_m - \xi) \cdot u_m d\xi dx ds. \end{aligned} \quad (3.47)$$

Combining (3.44) and (3.47), we have (3.42) for any $t \in [0, T_m]$. □

Note that all norms in X_m are equivalent, Lemma 3.7 yields a uniform estimate on u_m as follows

$$\sup_{0 \leq t \leq T_m} \|u_m\|_{X_m} \leq C(m) < \infty.$$

It allows us to have $T_m = T$. Thus, the solution (u_m, f_m) exists on all time t . And hence, we have the following result on the global existence of weak solutions of the approximation system:

Proposition 3.1. *For any $T > 0$, under the assumption of Theorem 1.1, there exists a weak solution (u_m, f_m) of the following problem*

$$\begin{cases} \partial_t f_m + \xi \cdot \nabla_x f_m + \operatorname{div}_\xi((u_m - \xi)f_\epsilon) = -\lambda f_m + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi', \\ \partial_t u_m + (u_m \cdot \nabla_x) u_m + \nabla_x P_m - \Delta_x u_m = - \int_{\mathbb{R}^3} (u_m - \xi) f_m d\xi, \\ \operatorname{div} u_m = 0, \quad x \in \mathbb{T}^3, \xi \in \mathbb{R}^3, t \in (0, T), \end{cases} \quad (3.48)$$

with initial data

$$(f_m, u_m)|_{t=0} = (f_0^\epsilon(x, \xi), u_0^\epsilon(x)), \quad x \in \mathbb{T}^3, \xi \in \mathbb{R}^3. \quad (3.49)$$

Moreover, for any $0 \leq t \leq T$, the weak solution (u_m, f_m) satisfies the following energy inequality

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla_x u_m|^2 dx ds \\ & + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds \\ & \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(1 + \frac{1}{2}|\xi|^2) d\xi dx, \end{aligned} \quad (3.50)$$

and

$$\|f_m(t, x, \xi)\|_{L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{T}^3))} \leq C, \quad \text{for any } 1 \leq p \leq +\infty, \quad (3.51)$$

where $C > 0$ depends only on the initial data, λ and T .

4 Recover the weak solutions

The main goal of this section is to recover the weak solutions of problem (1.7)-(1.8) by passing to the limits of (u_m, f_m) which constructed in Proposition 3.1. In particular, we shall pass to the limits as m goes to infinity and ϵ tends to zero, and show that the limit function is a weak solution of problem (1.7)-(1.8). In the following, we will investigate the weak limits with respect to m in Step 1 and pass to the limits with respect to ϵ in Step 2.

Step 1. Passing to the limit as $m \rightarrow \infty$.

In this step, we keep $\epsilon > 0$ fixed, deducing from Proposition 3.1, we have the following estimates independent of m :

$$\|f_m\|_{L^\infty(0,T;L^p(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \text{ for any } 1 \leq p \leq +\infty, \quad (4.1)$$

$$M_2 f_m(t) \leq C, \text{ for any } 0 \leq t \leq T, \quad (4.2)$$

$$\|u_m\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq C, \quad (4.3)$$

$$\|u_m\|_{L^2(0,T;H^1(\mathbb{T}^3))} \leq C. \quad (4.4)$$

Meanwhile, as the same in Section 3, the solution satisfies the following estimate:

$$\left\| \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi' \right\|_{L^\infty(0,T;L^p(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \text{ for any } 1 \leq p \leq +\infty. \quad (4.5)$$

Furthermore, by (4.1)-(4.4) and Lemma 2.3 and Lemma 3.4, any solution holds the following uniform bounds

$$M_3 f_m(t) \leq C, \text{ for any } 0 \leq t \leq T, \quad (4.6)$$

$$\left\| \int_{\mathbb{R}^3} f_m d\xi \right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq C, \quad (4.7)$$

$$\left\| \int_{\mathbb{R}^3} \xi f_m d\xi \right\|_{L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^3))} \leq C, \quad (4.8)$$

and

$$\left\| \int_{\mathbb{R}^3} |\xi|^2 f_m d\xi \right\|_{L^\infty(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))} \leq C. \quad (4.9)$$

With the above estimates (4.1)-(4.9), we are ready to investigate the limits as m goes to large. To this end, we shall rely on the Aubin-Lions lemma for the Navier-Stokes part and the L^p average velocity lemma for the kinetic part. Using the same arguments as that in [3], we prove that $\partial_t u_m$ is bounded in $L^{\frac{4}{3}}(0,T;H^{-1}(\mathbb{T}^3))$. For the completeness, we give the proof as follows.

Lemma 4.1. *For any m , it holds that*

$$\|\partial_t u_m\|_{L^{\frac{4}{3}}(0,T;H^{-1}(\mathbb{T}^3))} \leq C. \quad (4.10)$$

Proof. Note that

$$\partial_t u_m = -(u_m \cdot \nabla_x) u_m - \nabla_x P_m + \Delta_x u_m + \int_{\mathbb{R}^3} (u_m - \xi) f_m d\xi,$$

we control the first term

$$\int_0^T \int_{\mathbb{T}^3} (u_m \cdot \nabla) u_m \cdot \varphi dx ds = - \int_0^T \int_{\mathbb{T}^3} (u_m \cdot \nabla) \varphi \cdot u_m dx ds. \quad (4.11)$$

By (4.3)-(4.4) and interpolation inequality, one obtains

$$\|u_m\|_{L^{\frac{8}{3}}(0,T;L^4(\mathbb{T}^3))} \leq C. \quad (4.12)$$

From (4.11) and (4.12), we deduce that

$$\left| \int_0^T \int_{\mathbb{T}^3} (u_m \cdot \nabla) u_m \cdot \varphi dx ds \right| \leq C \|\nabla \varphi\|_{L^4(0,T;L^2(\mathbb{T}^3))}. \quad (4.13)$$

The second term vanishes if $\operatorname{div} \varphi = 0$, and the third term is controlled by

$$\int_0^T \int_{\mathbb{T}^3} \Delta u_m \varphi dx dt = - \int_0^T \int_{\mathbb{T}^3} \nabla u_m : \nabla \varphi dx dt,$$

which is bounded by $C \|\nabla \varphi\|_{L^2(0,T;L^2(\mathbb{T}^3))}$.

Next, for the last term, by Hölder inequality, we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(u_m - \xi) \cdot \varphi d\xi dx ds \right| \\ & \leq C \|u_m\|_{L^2(0,T;L^6(\mathbb{T}^3))} \|\varphi\|_{L^2(0,T;L^6(\mathbb{T}^3))} \|m_0 f_m\|_{L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^3))} \\ & \quad + C \|\varphi\|_{L^2(0,T;L^5(\mathbb{T}^3))} \|m_1 f_m\|_{L^2(0,T;L^{\frac{5}{4}}(\mathbb{T}^3))}, \end{aligned} \quad (4.14)$$

which implies $\int_{\mathbb{R}^3} f_m(u_m - \xi) d\xi$ is uniformly bounded in $L^2(0,T;H^{-1})$. This completes the proof of Lemma 4.1. □

With (4.4) and Lemma 4.1, the Aubin-Lions Lemma yields

$$u_m \rightarrow u \quad \text{strongly in } L^\infty(0,T;L^r(\mathbb{T}^3)) \quad (4.15)$$

for any $1 < r \leq 6$.

By (4.1), (4.3)-(4.4), as $m \rightarrow \infty$, we have

$$\begin{aligned} f_m & \rightharpoonup f, \text{ weakly(*) in } L^\infty(0,T;L^q(\mathbb{T}^3 \times \mathbb{R}^3)), \text{ for any } q > 1; \\ u_m & \rightharpoonup u, \text{ weakly(*) in } L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3)). \end{aligned} \quad (4.16)$$

Thanks to (4.5) and (4.16)₁, we employ the same arguments as in (3.26), to have, for any $q > 1$, as $m \rightarrow \infty$,

$$\int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') d\xi' \rightharpoonup \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') d\xi', \text{ weakly(*) in } L^\infty(0,T;L^q(\mathbb{T}^3 \times \mathbb{R}^3)). \quad (4.17)$$

Furthermore, by (4.7)-(4.9) and (4.16)₁, we have

$$\int_{\mathbb{R}^3} f_m d\xi \rightharpoonup \int_{\mathbb{R}^3} f d\xi, \text{ weakly(*) in } L^\infty(0, T; L^2(\mathbb{T}^3)), \quad (4.18)$$

$$\int_{\mathbb{R}^3} \xi f_m d\xi \rightharpoonup \int_{\mathbb{R}^3} \xi f d\xi, \text{ weakly(*) in } L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{T}^3)), \quad (4.19)$$

and

$$\int_{\mathbb{R}^3} |\xi|^2 f_m d\xi \rightharpoonup \int_{\mathbb{R}^3} |\xi|^2 f d\xi, \text{ weakly(*) in } L^\infty(0, T; L^{\frac{6}{5}}(\mathbb{T}^3)), \quad (4.20)$$

as $m \rightarrow \infty$.

Thus, we are ready to investigate the limits as m goes to large. In particular, with (4.15)-(4.16), as $m \rightarrow \infty$, from the following weak formulations:

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^3} (-u_m \cdot \partial_t \varphi + (u_m \cdot \nabla) u_m \cdot \varphi + \nabla u_m : \nabla \varphi) dx ds \\ &= - \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m (u_m - \xi) \cdot \varphi d\xi dx ds + \int_{\mathbb{T}^3} u_0^\epsilon \cdot \varphi(0, \cdot) dx, \end{aligned} \quad (4.21)$$

we can recover the following ones

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^3} (-u \cdot \partial_t \varphi + (u \cdot \nabla) u \cdot \varphi + \nabla u : \nabla \varphi) dx ds \\ &= - \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f (u - \xi) \cdot \varphi d\xi dx ds + \int_{\mathbb{T}^3} u_0^\epsilon \cdot \varphi(0, \cdot) dx, \end{aligned} \quad (4.22)$$

for any test function $\varphi \in C^\infty([0, T] \times \mathbb{T}^3)$ and $\operatorname{div} \varphi = 0$.

And similarly, thanks to (4.15)-(4.19), as $m \rightarrow \infty$, from the weak formulation of kinetic equation

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m (\phi_t + \xi \cdot \nabla_x \phi + (u_m - \xi) \cdot \nabla_\xi \phi) d\xi dx ds \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon \phi(0, x, \xi) d\xi dx + \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m \phi d\xi dx ds \\ & \quad - \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_m(t, x, \xi') \phi d\xi d\xi' dx ds, \end{aligned} \quad (4.23)$$

we can recover

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f (\phi_t + \xi \cdot \nabla_x \phi + (u - \xi) \cdot \nabla_\xi \phi) d\xi dx ds \\ &= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon \phi(0, x, \xi) d\xi dx + \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f \phi d\xi dx ds \\ & \quad - \lambda \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \phi d\xi d\xi' dx ds, \end{aligned} \quad (4.24)$$

for any test function $\phi \in C^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$.

The last task of this step is to study the limit of the energy inequality as m goes to large. In particular, we can state this limit in the following lemma.

Lemma 4.2. *The following energy inequality holds as $m \rightarrow \infty$:*

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla_x u|^2 dx ds \\ & + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f|u - \xi|^2 d\xi dx ds \\ & \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(1 + \frac{1}{2}|\xi|^2) d\xi dx. \end{aligned} \quad (4.25)$$

Proof. Here, we only focus on the most challenge term

$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds, \quad (4.26)$$

when studying the limit of

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u_m|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m(1 + \frac{1}{2}|\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla_x u_m|^2 dx ds \\ & + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds \\ & \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon(1 + \frac{1}{2}|\xi|^2) d\xi dx, \end{aligned}$$

as $m \rightarrow \infty$. To that purpose, the same as in [8], we rewrite (4.26) as follows

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds &= \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m|^2 d\xi dx ds \\ & - 2 \int_0^T \int_{\mathbb{T}^3} u_m \cdot \int_{\mathbb{R}^3} f_m \xi d\xi dx ds + \int_0^T \int_{\mathbb{T}^3} m_2 f_m dx ds. \end{aligned} \quad (4.27)$$

By (4.16)₁ ($f_m \rightharpoonup f$, weakly* in $L^\infty(0, T; L^\infty(\mathbb{T}^3 \times \mathbb{R}^3))$) and (4.15), the Fatou's Lemma yields

$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |u|^2 f \mathbf{1}_{|\xi| \leq l} d\xi dx ds \leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m|^2 d\xi dx ds,$$

where $l > 0$ be any positive number. Letting $l \rightarrow \infty$, we can apply the monotone convergence theorem to obtain

$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |u|^2 f d\xi dx ds \leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m|^2 d\xi dx ds. \quad (4.28)$$

Thanks to (4.15) and (4.19), we have

$$\int_0^T \int_{\mathbb{T}^3} u_m \cdot \int_{\mathbb{R}^3} f_m \xi d\xi dx ds \rightarrow \int_0^T \int_{\mathbb{T}^3} u \cdot \int_{\mathbb{R}^3} f \xi d\xi dx ds, \text{ as } m \rightarrow \infty. \quad (4.29)$$

Note that (4.20), the Fatou's lemma allows us to have

$$\int_0^T \int_{\mathbb{T}^3} m_2 f dx ds \leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^3} m_2 f_m dx ds, \quad (4.30)$$

By (4.28)-(4.30), we have

$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f |u - \xi|^2 d\xi dx ds \leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_m |u_m - \xi|^2 d\xi dx ds, \text{ as } m \rightarrow \infty. \quad (4.31)$$

Thus, we are able to show (4.25). \square

By (4.22), (4.24), and Lemma 4.2, we have the following existence result of weak solutions for all time $t > 0$:

Proposition 4.1. *For any $T > 0$, under the assumption of Theorem 1.1, there exists a weak solution (u_ϵ, f_ϵ) of the following problem*

$$\begin{cases} \partial_t f_\epsilon + \xi \cdot \nabla_x f_\epsilon + \operatorname{div}_\xi((u_\epsilon - \xi)f_\epsilon) = -\lambda f_\epsilon + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_\epsilon(t, x, \xi') d\xi', \\ \partial_t u_\epsilon + (u_\epsilon \cdot \nabla_x) u_\epsilon + \nabla_x P_\epsilon - \Delta_x u_\epsilon = - \int_{\mathbb{R}^3} (u_\epsilon - \xi) f_\epsilon d\xi, \\ \operatorname{div} u_\epsilon = 0, \quad x \in \mathbb{T}^3, \quad \xi \in \mathbb{R}^3, \quad t \in (0, T), \end{cases} \quad (4.32)$$

with initial data

$$(f_\epsilon, u_\epsilon)|_{t=0} = (f_0^\epsilon(x, \xi), u_0^\epsilon(x)), \quad x \in \mathbb{T}^3, \quad \xi \in \mathbb{R}^3. \quad (4.33)$$

Moreover, for any $0 \leq t \leq T$, the weak solution (u_ϵ, f_ϵ) satisfies the following energy inequality

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u_\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla_x u_\epsilon|^2 dx ds \\ & + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_\epsilon |u_\epsilon - \xi|^2 d\xi dx ds \\ & \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx, \end{aligned} \quad (4.34)$$

and

$$\|f_\epsilon(t, x, \xi)\|_{L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{T}^3))} \leq C, \quad \text{for any } 1 \leq p \leq +\infty, \quad (4.35)$$

here $C > 0$ depends only on the initial data, λ and T .

Step 2. Passing to the limit as $\epsilon \rightarrow 0$.

Next, we aim to pass to the limits for recovering the weak solution to the problem (1.7)-(1.8) as $\epsilon \rightarrow 0$.

Remark 4.1. *Although we fixed $\epsilon > 0$ in Step 1, estimates (4.1)-(4.9) are uniformly with respect to ϵ . Thus, we have the same convergence of (u_ϵ, f_ϵ) as the same to (u_m, f_m) .*

To recover the weak solutions of (1.7)-(1.8), we only need to pass to the limits of (u_ϵ, f_ϵ) as $\epsilon \rightarrow 0$. Thanks to Remark 4.1, we can take the same limit process as $m \rightarrow \infty$ to handle the limits in the weak formulation. It allows us to have (1.13), (1.14) by letting ϵ goes to zero.

Note that the restriction on u_0^ϵ and f_0^ϵ at the beginning of Section 3, we have

$$\frac{1}{2} \int_{\mathbb{T}^3} |u_0^\epsilon|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0^\epsilon (1 + \frac{1}{2} |\xi|^2) d\xi dx \rightarrow \frac{1}{2} \int_{\mathbb{T}^3} |u_0|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0 (1 + \frac{1}{2} |\xi|^2) d\xi dx$$

as $\epsilon \rightarrow 0$. With the help of the convergence of (u_ϵ, f_ϵ) , this allows us to have, as $\epsilon \rightarrow 0$,

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^3} |u|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f (1 + \frac{1}{2} |\xi|^2) d\xi dx + \int_0^T \int_{\mathbb{T}^3} |\nabla_x u|^2 dx ds \\ & + \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f |u - \xi|^2 d\xi dx ds \\ & \leq \frac{1}{2} \int_{\mathbb{T}^3} |u_0|^2 dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_0 (1 + \frac{1}{2} |\xi|^2) d\xi dx. \end{aligned} \quad (4.36)$$

Thus, we complete the proof of our main result.

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